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SOME EXTENSION THEOREMS AND COMPACTNESS CRITERIA IN ANALYSIS

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CHAPTER I

INTRODUCTION

The purpose of this thesis is to give a reasonably self-contained and detailed exposition of some criteria for compactness or relative compactness of subsets of various specific metric spaces, and to give detailed proofs of two forms of the Tietze extension theorem in a metric space, forms commonly used in analysis.

If X is a non-empty set, then a *metric* on X is a function d which assigns to each ordered pair (x,y) of elements of X a non-negative real number d(x,y) such that:

(a) $d(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$,

(b) $d(x,y) = d(y,x)$, and

(c) $d(x,y) \leq d(x,z) + d(z,y)$.

A *metric space* consists of a non-empty set X and a metric d on X. As an example, note that the set of all real numbers with $d(x,y) = |x - y|$ forms the metric space R^1.

Chapter II is concerned with the notion of compactness in a metric space, and various ways of describing compact or relatively compact sets. These notions are then applied to two specific metric spaces: $C^r(X)$, the Banach space of real-valued continuous functions on a compact metric space X, with the uniform norm, and $L^p(m)(1 \leq p < \infty)$, the Banach space associated with the class of Lebesgue measurable functions on the real line which are pth power integrable in the Lebesgue sense. In the case of $C^r(X)$, the Arzelà-Ascoli theorem is proved. In the case of $L^p(m)$, the Riesz-Kolmogorov theorem is established. The proof here follows the lines of one given by K. Yosida, but is given here in an expanded form. This theorem is a very important one in real analysis, and has led to several generalizations in the literature. Detailed references are given in the thesis.

Chapter III includes proofs of two related versions of Tietze's extension theorem in a metric space. The arguments are slightly modified versions of proofs by McShane and Botts, and Dieudonné. Similar proofs occur elsewhere in the literature, and the methods are based wholly on classical analysis. The Tietze extension theorem is sometimes considered as a first step in the study of the general problem of extending a continuous mapping of a closed subset D of a space X into a space Y, to a continuous mapping of the whole space X into Y.

References to the Bibliography are in the form of the author's name followed in a square bracket by the number of the reference in the Bibliography. The end of a proof will be indicated by the symbol $\square$.

CHAPTER II

COMPACTNESS IN METRIC SPACES

In the Euclidean spaces R^n, the compact sets are precisely the sets which are both closed and bounded. In more general metric spaces, closed and bounded sets need not be compact. The following example will illustrate this assertion in the metric space $C^r[0,1]$ of all real-valued continuous functions on the interval $[0,1]$, with the metric defined by

$$d(f,g) = \max_{0 \le x \le 1} |f(x) - g(x)| .$$

This metric space is a complete metric space, and if we define

$$\|f\| = \max_{0 \le x \le 1} |f(x)|$$

then $\|f\|$ is a norm on $C^r[0,1]$ and $d(f,g) = \|f - g\|$. It follows that $C^r[0,1]$ is then a complete normed linear space, or a *Banach space.*

As an example, consider the subset E of $C^r[0,1]$ defined by

$$E = \{f : f \varepsilon C^r[0,1],\ f(0) = 0,\ f(1) = 1, \text{ and } \max_{0 \le x \le 1} |f(x)| \le 1\}$$

and the functional G defined on $C^r[0,1]$ by

$$G(f) = \int_0^1 (f(x))^2 dx \ .$$

If f and f_o are elements of E, note that

$$\begin{aligned} |G(f) - G(f_o)| &= \left|\int_0^1 [(f(x))^2 - (f_o(x))^2]dx\right| \\ &\leq \int_0^1 |f(x) - f_o(x)|\big(|f(x)| + |f_o(x)|\big)dx \\ &\leq 2\int_0^1 |f(x) - f_o(x)|dx \\ &\leq 2d(f,f_o) \ . \end{aligned}$$

It follows that the real-valued functional G is continuous on E. Note that if $f_n(x) = x^n (n = 1,2,\cdots)$, then each $f_n \varepsilon E$ and

$$G(f_n) = \int_0^1 x^{2n} dx = \frac{1}{2n+1} \ .$$

This implies that

$$\inf\{G(f) : f\varepsilon E\} = 0 \ .$$

Since f is real-valued and continuous on [0,1],

$$\int_0^1 (f(x))^2 dx = 0$$

if and only if $f(x) = 0$ for all x in $[0,1]$. However, the identically zero function on $[0,1]$ does not belong to E. This implies that E, although a closed and bounded set in $C^r[0,1]$, is not a compact set in $C^r[0,1]$.

An interesting and extensive article on the role of compactness in analysis is that of Hewitt [5].

Definition 2.1. A metric space X is *compact* if and only if each open cover $\{V_\alpha : \alpha \varepsilon I\}$ of the space X contains a finite subcover.

Definition 2.2. A metric space X is said to be *complete* if each Cauchy sequence in X converges to a point of X.

Definition 2.3. A metric space X is said to be *totally bounded* if for every $\varepsilon > 0$ there exists a finite subset $\{x_1,\cdots,x_n\}$ of X such that the collection $\{B(x_i,\varepsilon) : i = 1,2,\cdots,n\}$ of open spherical neighborhoods (each of radius ε) covers X.

Definition 2.4. A finite subset $\{x_1,x_2,\cdots,x_n\}$ of X such that $X \subset \bigcup_{i=1}^{n} B(x_i,\varepsilon)$ is called an ε-*net* in the metric space X.

Thus, a metric space X is totally bounded if for every $\varepsilon > 0$ there exists an ε-net in X.

Theorem 2.5. Let X be a metric space with distance function d. The following assertions are equivalent:

(1) X is compact.

(2) Every infinite subset of X has a limit point in X.

(3) Every sequence of points of X has a subsequence convergent to some point in X.

(4) X is complete and totally bounded.

Proof. Assume that (1) is true. Suppose that E is an infinite subset of X with no limit point in X. For each $x \varepsilon E^c = X - E$, there exists a neighborhood U_x with $U_x \cap E = \phi$ and thus $U_x \subset E^c$. Thus E^c is an open set. For each point $x\varepsilon E$, there exists a neighborhood V_x such that $V_x \cap E = \{x\}$ since x is not a limit point of E. Let such a neighborhood V_x be considered for each $x\varepsilon E$. Then the collection

$$\{V_x : x\varepsilon E\} \cup \{E^c\}$$

is an open cover of X with no finite subcover. This contradicts the assumed compactness of X. Thus (1) implies (2).

Now assume that (2) is true. Let $\{x_n\}$ be a sequence of points of X. If the range of $\{x_n\}$ is a finite set, there exists a sequence $\{n_k\}$ of indices with $n_1 < n_2 < \cdots < n_k < \cdots$ such that $x_{n_1} = x_{n_2} = \cdots = x_{n_k} = \cdots$. In this case,

$$\lim_{k\to\infty} x_{n_k} = x_{n_1}$$

and $\{x_{n_k}\}$ is a convergent subsequence of $\{x_n\}$. If, on the other hand, the range of $\{x_n\}$ is an infinite set, then by (2), the range has a limit point $x_o \varepsilon X$. Let n_1 be the smallest index such that $x_{n_1} \varepsilon B(x_o,1)$. Let n_2 be the smallest index such that $n_2 > n_1$ and $x_{n_2} \varepsilon B(x_o,\frac{1}{2})$. If $n_1, \cdots, n_k$

are determined, let n_{k+1} be the smallest index such that $n_{k+1} > n_k$ and $x_{n_{k+1}} \varepsilon B(x_o, \frac{1}{k+1})$. Thus a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is defined inductively, and in such a way that

$$d(x_{n_k}, x_o) < \frac{1}{k} \qquad (k = 1,2,\cdots) .$$

It follows that $\lim_{k\to\infty} x_{n_k} = x_o \varepsilon X$. Thus (3) is true, and it has been proved that (2) implies (3).

Now, assume that (3) is true. Suppose $\{x_n\}$ is a Cauchy sequence in X. By (3), there exists a subsequence $\{x_{n_k}\}$ convergent to a point $x_o \varepsilon X$. Given $\varepsilon > 0$, there exists an integer $N \geq 1$ such that $d(x_m, x_n) < \frac{\varepsilon}{2}$ whenever $m > N$ and $n > N$. Since $\lim_{k\to\infty} x_{n_k} = x_o$, there exists an index i such that $n_i > N$ and $d(x_{n_i}, x_o) < \frac{\varepsilon}{2}$. Thus, if $n > N$,

$$d(x_n, x_o) \leq d(x_n, x_{n_i}) + d(x_{n_i}, x_o) < \varepsilon ,$$

and thus $\lim_{n\to\infty} x_n = x_o$. Since every Cauchy sequence of points of X converges to a point of X, X is a complete metric space.

Now, suppose that X is not totally bounded. Then there exists $\varepsilon > 0$ such that no finite collection of open spherical neighborhoods, each of radius ε, covers X. Consider such a number ε, and suppose $x_1 \varepsilon X$. Then $B(x_1, \varepsilon)$ does not cover X. Thus there exists a point $x_2 \notin B(x_1, \varepsilon)$. Then $B(x_1, \varepsilon) \cup B(x_2, \varepsilon)$ also fails to cover X, and thus there exists a point $x_3 \notin B(x_1, \varepsilon) \cup B(x_2, \varepsilon)$. Having obtained $x_1, \cdots, x_{k-1}$, there exists a point $x_k \notin B(x_1, \varepsilon) \cup \cdots \cup B(x_{k-1}, \varepsilon)$. Hence there exists a sequence $\{x_n\}$ of points of X such that

$$x_k \notin \bigcup_{i=1}^{k-1} B(x_i,\varepsilon) \qquad (k = 2,3,\cdots) .$$

If $m \neq n$, it follows that

$$d(x_m,x_n) \geq \varepsilon .$$

Otherwise, if $m < n$, $x_n \in B(x_m,\varepsilon) \subset \bigcup_{i=1}^{n-1} B(x_i,\varepsilon)$, which is false by the construction of $\{x_k\}$. Thus no subsequence of $\{x_n\}$ can be a Cauchy sequence and thus certainly no subsequence of $\{x_n\}$ is convergent to a point of X. This contradicts (3). It is thus proved that, assuming (3), X is necessarily complete and totally bounded. Hence (3) implies (4).

Finally, assume that (4) is true. Let $\{V_\alpha : \alpha\varepsilon I\}$ be a collection of open sets in X for which no finite subcollection covers X. It will be shown that there exists a point $x_o \varepsilon X$ such that $x_o \notin \bigcup_{\alpha\varepsilon I} V_\alpha$. This implies that if $\{V_\alpha : \alpha\varepsilon I\}$ is an open cover of X, then some finite subcollection of $\{V_\alpha : \alpha\varepsilon I\}$ necessarily covers X, and thus X is compact. Since X is totally bounded if (4) is assumed, there exists an ε-net for each $\varepsilon > 0$. Since X has a 1-net, there exists an open spherical neighborhood U_1 of radius 1 and center at a point of this 1-net such that U_1 is not covered by any finite subcollection of the sets $\{V_\alpha : \alpha\varepsilon I\}$. Otherwise, X could be covered by a finite subcollection of $\{V_\alpha : \alpha\varepsilon I\}$. Again using the fact that X is totally bounded, X has a $\frac{1}{2}$-net. Consider the (finite number of) open spherical neighborhoods of radius $\frac{1}{2}$ centered at points of the $\frac{1}{2}$-net which contain points of U_1.

At least one of these neighborhoods, U_2, is not covered by any finite subcollection of the sets $\{V_\alpha : \alpha \varepsilon I\}$. Otherwise, U_1 would be covered by a finite subcollection of $\{V_\alpha : \alpha \varepsilon I\}$. Suppose that $U_1, \cdots, U_i$ have been obtained, with $U_{i-1} \cap U_i \neq \phi$, U_i of radius $\frac{1}{2^{i-1}}$, and U_i not covered by any finite subcollection of the sets $\{V_\alpha : \alpha \varepsilon I\}$. Since X is totally bounded, X has a $\frac{1}{2^i}$ - net. Consider the (finite number of) open spherical neighborhoods of radius 2^{-i} centered at points of this $\frac{1}{2^i}$-net which contain points of U_i. At least one of these neighborhoods, U_{i+1}, is not covered by any finite subcollection of $\{V_\alpha : \alpha \varepsilon I\}$, for otherwise U_i would be so covered. Thus there exists a sequence $\{U_i\}$ of open spherical neighborhoods such that U_i has radius $\frac{1}{2^{i-1}}$, $U_i \cap U_{i+1} \neq \phi$ $(i = 1,2,\cdots)$, and no U_i is covered by any finite subcollection of $\{V_\alpha : \alpha \varepsilon I\}$. Let x_i be the center of U_i. If $\bar{x}_i \; \varepsilon$ $U_i \cap U_{i+1}$, it follows that

$$d(x_i, x_{i+1}) \leq d(x_i, \bar{x}_i) + d(\bar{x}_i, x_{i+1})$$

$$< \frac{1}{2^{i-1}} + \frac{1}{2^i} < \frac{1}{2^{i-2}} .$$

If $m < n$,

$$d(x_m, x_n) \leq d(x_m, x_{m+1}) + \cdots + d(x_{n-1}, x_n)$$

$$< \frac{1}{2^{m-2}} + \cdots + \frac{1}{2^{n-3}} < \frac{1}{2^{m-1}} .$$

Hence $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists an $x_o \in X$ such that $x_n \to x_o$ as $n \to \infty$. It will now be shown that $x_o \notin \bigcup_{\alpha \in I} V_\alpha$. Suppose $x_o \in \bigcup_{\alpha \in I} V_\alpha$, then there exists an index $\beta \in I$ such that $x_o \in V_\beta$. Since V_β is open, there exists an open spherical neighborhood $B(x_o,\varepsilon) \subset V_\beta$. Since $x_n \to x_o$ and x_n is the center of U_n, there exists $N \geq 1$ such that $d(x_N,x_o) < \frac{\varepsilon}{2}$. Since U_n has radius $\frac{1}{2^{n-1}}$, N can be assumed chosen sufficiently large so that $\frac{1}{2^{N-1}} < \frac{\varepsilon}{2}$. Then if $x \in U_N$,

$$d(x,x_o) \leq d(x,x_N) + d(x_N,x_o) < \varepsilon .$$

This inequality implies that $U_N \subset B(x_o,\varepsilon) \subset V_\beta$, i.e., that one set of $\{V_\alpha : \alpha \in I\}$ covers U_N. This is a contradiction, since no U_n is so covered, by construction of the sequence $\{U_n\}$. It follows that $x_o \notin \bigcup_{\alpha \in I} V_\alpha$. Referring to the argument at the beginning of this part of the proof, it follows that X is compact. Hence (4) implies (1). ▯

Definition 2.6. If (X,d) is a metric space and $E \subset X$, then E is a *compact set* in X if (E,d), regarded as a metric space, is compact.

Definition 2.7. If $A \subset E \subset X$, $a \in A$, and E is regarded as the metric space containing A, then the open spherical neighborhood $B_E(a,r)$ is the set

$$\{x : x \in E,\ d(a,x) < r\} = B(a,r) \cap E,$$

where

$$B(a,r) = \{x : x \varepsilon X,\ d(a,x) < r\} .$$

Definition 2.8. A set $A \subset E$ is *open in E* (open relative to E) if, for each $a \varepsilon A$, there exists an $r(a) > 0$ such that $B_E(a,r(a)) \subset A$.

Theorem 2.9. A is open in E if and only if there exists an open set V in X such that $A = E \cap V$.

Proof. Suppose that A is open in E. If $A \neq \phi$, then there exists a collection of number $\{r(a) : a \varepsilon A\}$ such that $\bigcup_{a \varepsilon A} B_E(a,r(a)) \subset A$, and hence

$$E \cap (\bigcup_{a \varepsilon A} B(a,r(a)) \subset A .$$

On the other hand, since $A \subset E$ and $A \subset \bigcup_{a \varepsilon A} B(a,r(a))$ then

$$A \subset E \cap \left(\bigcup_{a \varepsilon A} B(a,r(a))\right).$$

Hence

$$A = E \cap V$$

where

$$V = \bigcup_{a \varepsilon A} B(a,r(a))$$

is open in X.

Now suppose $A = E \cap V$, where V is open in X. If $a \varepsilon A$, then $a \varepsilon E$ and $a \varepsilon V$. Thus there exists a neighborhood $B(a,r(a)) \subset V$, and hence

$$E \cap B(a,r(a)) \subset E \cap V,$$

and thus

$$B_E(a,r(a)) \subset E \cap V = A,$$

which is the same as saying that A is open in E. ▯

Theorem 2.10. Let X be a metric space and $A \subset E \subset X$. A is closed in E if and only if $A = E \cap F$ where F is a set closed in X.

This theorem is analogous to Theorem 2.9, and the proof is omitted here.

Theorem 2.11. Suppose X is a metric space and $E \subset X$. Then E is a compact set in X if and only if every cover of E by a collection of open sets in X has a finite subcover.

Proof. Suppose that every cover of E by a collection of open sets in X has a finite subcover. Suppose that $\bigcup_\alpha W_\alpha \supset E$, with each W_α open in E. By Theorem 2.9 there exists a set V_α open in X such that $W_\alpha = E \cap V_\alpha$. It follows that $\bigcup_\alpha V_\alpha \supset E$. Thus there exists a finite subcollection $\{V_{\alpha_1},\cdots,V_{\alpha_n}\}$ such that

$$\bigcup_{i=1}^{n} V_{\alpha_i} \supset E .$$

Then

$$\bigcup_{i=1}^{n} W_{\alpha_i} = \bigcup_{i=1}^{n} (E \cap V_{\alpha_i}) \supset E ,$$

and it has been shown that a finite subcollection of $\{W_\alpha\}$ covers E. Thus E is a compact set in X by Definition 2.6.

Now suppose that E is a compact set in X and that

$$\bigcup_{\alpha} V_\alpha \supset E ,$$

with each V_α open in X. It follows that

$$\bigcup_{\alpha} (E \cap V_\alpha) \supset E ,$$

where each set $E \cap V_\alpha$ is open in E by Theorem 2.9. Thus there exists a finite subcollection $\{E \cap V_{\alpha_1}, \cdots, E \cap V_{\alpha_k}\}$ such that

$$\bigcup_{i=1}^{k} (E \cap V_{\alpha_i}) \supset E .$$

Thus $\bigcup_{i=1}^{k} V_{\alpha_i} \supset E$ and it has been proved that every cover of E by a collection of open sets in X has a finite subcover.

Theorem 2.12. Let X be a complete metric space, and suppose $E \subset X$. The closure $\bar{E}$ of E is compact if and only if E is totally bounded.

Proof. Note first that $\bar{E}$ is a closed set in X. Thus $\bar{E}$, regarded as a metric space, is a complete metric space. Now suppose that E is totally bounded. Then for each $\varepsilon > 0$ there exists an $\frac{\varepsilon}{2}$—net $\{x_1, \cdots, x_n\}$ in E such that

$$\bigcup_{i=1}^{n} B(x_i, \tfrac{\varepsilon}{2}) \supset E \ .$$

It follows that

$$\bigcup_{i=1}^{n} \overline{B(x_i, \tfrac{\varepsilon}{2})} \supset E \ ,$$

and hence, since the set $\bigcup_{i=1}^{n} \overline{B(x_i, \tfrac{\varepsilon}{2})}$ is a finite union of closed sets and is thus a closed set,

$$\bigcup_{i=1}^{n} \overline{B(x_i, \tfrac{\varepsilon}{2})} \supset \bar{E} \ .$$

Hence

$$\bigcup_{i=1}^{n} B(x_i, \varepsilon) \supset \bar{E} \ ,$$

and thus $\bar{E}$ is totally bounded. Since $\bar{E}$ is, as a metric space, complete and totally bounded, it is a consequence of Theorem 2.5 that $\bar{E}$ is compact.

Now suppose that $\bar{E}$ is compact. Then by Theorem 2.5 $\bar{E}$ is totally bounded, and it follows easily that E is totally bounded. $\square$

Definition 2.13. A subset E of a metric space X is said to be *relatively compact* if its closure $\bar{E}$ is compact.

In view of Definition 2.13, Theorem 2.12 asserts that, if X is a complete metric space, then a subset $E \subset X$ is relatively compact if and only if E is totally bounded.

If X is a metric space, and

$$B(a,r) = \{x : d(x,a) < r\}$$

$$B'(a,r) = \{x : d(x,a) \leq r\} ,$$

the set $B'(a,r)$ is a closed set in X (since its complement is easily shown to be open in X). It should be noted that $\overline{B(a,r)}$, the closure of $B(a,r)$ need not be the same as $B'(a,r)$. In all cases it is true that

$$(1) \quad \overline{B(a,r)} \subset B'(a,r)$$

since $\overline{B(a,r)}$ is the smallest closed set containing $B(a,r)$ and $B'(a,r)$ is a closed set containing $B(a,r)$. To show that equality need not hold in (1), consider the set of integers in R^1 with the R^1 metric. Then $B(0,1) = \{0\}$ and $\overline{B(0,1)} = \{0\}$, but $B'(0,1) = \{-1,0,1\}$.

Definition 2.14. A collection F of real or complex valued functions on a set X is said to be *uniformly bounded* if there is a real number $M > 0$ such that $|f(x)| \leq M$ for all $x \varepsilon X$ and all $f \varepsilon F$.

Definition 2.15. A collection F of functions defined on a metric space X is said to be *equicontinuous* if for each $\varepsilon > 0$ there is a $\delta > 0$ such that, if $x, x' \varepsilon X$ and $d(x,x') < \delta$, then $|f(x) - f(x')| < \varepsilon$ for all $f \varepsilon F$.

Definition 2.16. Let X be a compact metric space. $C^r(X)$ will denote the set of all real-valued continuous functions defined on X.

It follows easily that $C^r(X)$ is a metric space with the metric defined by

$$d(f,g) = \sup[|f(x) - g(x)| : x\varepsilon X] .$$

This metric space is a complete metric space, and if we define

$$\|f\|_\infty = \sup\{|f(x)| : x\varepsilon X\} ,$$

then $\|f\|_\infty$ is a norm on $C^r(X)$ and hence $C^r(X)$ is also a Banach space, i.e., a complete normed linear space.

Theorem 2.17 (Arzelà-Ascoli). If X is a compact metric space, a subset $K \subset C^r(X)$ is relatively compact if and only if it is uniformly bounded and equicontinuous.

Proof. Suppose K is relatively compact. Then K is totally bounded by Theorem 2.12. Let $\varepsilon > 0$ be given, and let $f_1, f_2, \cdots, f_n$ be an $\frac{\varepsilon}{3}$–net in K. Each of the functions $f_1, f_2, \cdots, f_n$ is a continuous real-valued function on X. Thus f_i $(i = 1,2,\cdots,n)$ is bounded (Hewitt [4], p. 74) and thus for each $i = 1,2,\cdots,n$, there exists a number $M_i > 0$ such that

$|f_i(x)| < M_i$ for all $x \varepsilon X$. Now let

$$M = \max\{M_i : i = 1,2,\cdots,n\} + \frac{\varepsilon}{3} .$$

By Definition 2.4 of an $\frac{\varepsilon}{3}$-net, for each $f \varepsilon K$ we can choose j such that

$$(1) \quad \|f - f_j\|_\infty = \sup[|f(x) - f_j(x)| : x \varepsilon X] < \frac{\varepsilon}{3} .$$

Thus, for every $x \varepsilon X$,

$$\begin{aligned} |f(x)| &= |f(x) - f_j(x) + f_j(x)| \\ &\le |f(x) - f_j(x)| + |f_j(x)| \\ &< \frac{\varepsilon}{3} + |f_j(x)| \\ &< M_j + \frac{\varepsilon}{3} \\ &< M , \end{aligned}$$

and hence $\|f\|_\infty < M$, and the set K is uniformly bounded.

Now let $f \varepsilon K$ and $x, x' \varepsilon X$. Then for each $i = 1,2,\cdots,n$,

$$(2) \quad |f(x) - f(x')| \le |f(x) - f_i(x)| + |f_i(x) - f_i(x')| + |f_i(x') - f(x')|,$$

From (1) and (2) we obtain, for some $j = 1,\cdots,n$,

$$(3) \quad |f(x) - f(x')| < |f_j(x) - f_j(x')| + \frac{2\varepsilon}{3} \ .$$

Since X is compact, the functions $f_i (i = 1,\cdots,n)$ are uniformly continuous. Thus there exists $\delta_i > 0$ such that $d(x,x') < \delta_i$ implies

$$(4) \quad |f_i(x) - f_i(x')| < \frac{\varepsilon}{3} \ (i = 1,2,\cdots,n) \ .$$

We now define δ to be the smallest of the numbers $\delta_1, \delta_2, \cdots, \delta_n$. Then from (2), (3) and (4) we see that if $x,x' \varepsilon X$ and $d(x,x') < \delta$, then

$$|f(x) - f(x')| < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon$$

for every $f \varepsilon K$. Thus K is equicontinuous, by Definition 2.15.

Now suppose, conversely, that K is uniformly bounded and equicontinuous. By Definition 2.14 there is an integer $M > 0$ such that

$$|f(x)| \leq M \quad \text{for all } x \varepsilon X \text{ and all } f \varepsilon K \ .$$

By Definition 2.15, for each $\varepsilon > 0$ there is a $\delta > 0$ such that $d(x,x') < \delta$ implies $|f(x) - f(x')| < \frac{\varepsilon}{4}$ for all $x,x' \varepsilon X$ and all $f \varepsilon K$. Since X is compact, then X is totally bounded by Theorem 2.5, and thus X has a δ-net$\{x_1,\cdots,x_n\}$. Choose a positive integer m such that $\frac{1}{m} < \frac{\varepsilon}{4}$ and divide $[-M,M]$ into $2Mm$ equal parts each of length $\frac{1}{m}$ by the points

$$y_o = -M < y_1 < y_2 < \cdots < y_k = M ,$$

where $k = 2Mm$.

Consider those n-tuples $(y_{i_1}, y_{i_2}, \cdots, y_{i_n})$ of the numbers y_i, $i = 0,1,\cdots,k$, such that some $f \varepsilon K$ has the property

$$|f(x_j) - y_{i_j}| < \frac{\varepsilon}{4} , \quad (j = 1,2,\cdots,n) .$$

Choose one such $f \varepsilon K$ for each n-tuple and call E the resulting finite subset of K. It will now be shown that E is an ε-net for K. If $f \varepsilon K$, then we may choose $y_{i_1}, y_{i_2}, \cdots, y_{i_n}$ so that

$$(5) \quad |f(x_j) - y_{i_j}| < \frac{\varepsilon}{4} , \quad (j = 1,2,\cdots,n) ,$$

and thus there is a corresponding $g \varepsilon E$ associated with $(y_{i_1}, y_{i_2}, \cdots, y_{i_n})$.

Let $x \varepsilon X$ and choose j so that $d(x,x_j) < \delta$. Then by (5), and the equicontinuity of K,

$$(6) \quad |f(x) - g(x)| \leq |f(x) - f(x_j)| + |f(x_j) - y_{i_j}|$$
$$+ |y_{i_j} - g(x_j)| + |g(x_j) - g(x)|$$
$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon .$$

The argument leading to (6) implies that E is an ε-net in K, and hence that K is totally bounded. Then $\bar{K}$ is compact by Theorem 2.12 and hence K is relatively compact. $\square$

We shall only sketch here the definition of $L^p(\mu)$, for $1 \leq p < \infty$, referring for proofs and further discussion to any standard reference in real analysis, as for example, Royden [9] or Hewitt and Stromberg [4]. Suppose $(X,\mathcal{S},\mu)$ is a measure space, with $\mathcal{S}$ a σ-algebra of subsets of X and μ a positive measure on $\mathcal{S}$. Consider all real-valued (or complex-valued) measurable functions f on X for which the Lebesgue integral

$$\int_X |f|^p \, d\mu < \infty \ .$$

This set of functions is a linear space, M^p. We say that $f \sim g$ if $f(x) = g(x)$ almost everywhere on X relative to the measure μ, and in this way obtain an equivalence relation in the set M^p of functions under consideration. This equivalence relation partitions M^p into disjoint equivalence classes. Thus, if $f \varepsilon M^p$, f belongs to (and determines) the equivalence class

$$\hat{f} = \{g : g\varepsilon M^p, \quad g \sim f\} \ .$$

We define $\hat{f} + \hat{g} = \widehat{f + g}$, and $c\hat{f} = \widehat{cf}$ for any scalar c. Then the linear space of all equivalence classes is called $L^p(\mu)$. For any $g \varepsilon f$,

$$\int_X |g|^p \, d\mu = \int_X |f|^p \, d\mu < \infty .$$

If we define

$$\|\hat{f}\|_p = \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}} ,$$

then

$$\|\hat{f}\|_p = 0 \quad \text{if and only if} \quad \hat{f} = \hat{0} ,$$

$$\|\hat{f} + \hat{g}\|_p \leq \|\hat{f}\|_p + \|\hat{g}\|_p ,$$

$$\|c\hat{f}\|_p = |c| \; \|\hat{f}\|_p .$$

We thus have defined a *norm* on $L^p(\mu)$, and with this norm $L^p(\mu)$ is a normed linear space. It can be proved that $L^p(\mu)$ is a complete normed linear space, i.e., a Banach space.

In actual writing, it is not customary to continue to use the notation adopted here. One speaks of f itself as an element of $L^p(\mu)$, rather than of $\hat{f}$. What is meant is that f determines an element of $L^p(\mu)$. Usually the language used in practice causes no confusion.

Lemma 2.18. Suppose g is continuous on R^1. If $g(x) = 0$ for all $x \notin [a,b]$ for some compact interval $[a,b]$, then g is uniformly continuous on R^1. In particular, if g is continuous on R^1 and of compact support, g is uniformly continuous on R^1.

Proof. Since g is continuous on R^1, and $g(x) = 0$ for $x < a$, then $g(a) = 0$. Similarly, $g(b) = 0$. The assertion now follows at once from the theorem which asserts that a continuous function on a compact set $[a,b]$ is uniformly continuous on $[a,b]$.

If g is continuous on R^1 and of compact support, then

$$\mathrm{supp}(g) = \overline{\{x : g(x) \neq 0\}}$$

is a compact set in R^1 and consequently is closed and bounded. Thus there is a compact interval $[a,b]$ containing the support of g, and $g(x)$ vanishes outside $[a,b]$. ▯

Lemma 2.19. Let $f \varepsilon L^P(m)$, where m is the usual Lebesgue measure on the Lebesgue measurable sets in R^1, and $1 \leq P < \infty$. Then

$$\lim_{h \to 0} \int_{-\infty}^{\infty} |f_h(x) - f(x)|^P \, dx = 0 \ ,$$

where $f_h(x) = f(x+h)$ for all real numbers x and h.

Proof. A standard theorem for this L^P space implies that, given $\varepsilon > 0$, there exists a continuous function g on R^1, of compact support, such that

$$(1) \quad \|f - g\|_p < \frac{\varepsilon}{3} \ .$$

The theorem asserts that the set $C_c(R^1)$ of continuous functions on R^1 with compact supports is dense in $L^P(m)$, and is proved in Hewitt and

Stromberg [4] (pp. 197-198), for instance. Using the fact that Lebesgue measure and measurability in R^1 are invariant under translation, another standard theorem asserts that $\|f_h\|_p = \|f\|_p$. By Minkowski's inequality, we have

$$(2)\quad \|f_h - f\|_p = \|f_h - g_h + g_h - g + g - f\|_p$$

$$\leq \|f_h - g_h\|_p + \|g_h - g\|_p + \|g - f\|_p \ .$$

Here,

$$(3)\quad \|f_h - g_h\|_p = \|f - g\|_p < \frac{\varepsilon}{3} \ ,$$

since $f_h - g_h = (f - g)_h$, and $\|f - g\|_p = \|(f - g)_h\|_p$, as remarked earlier in the proof. Thus (2) yields

$$(4)\quad \|f_h - f\|_p < \frac{2\varepsilon}{3} + \|g_h - g\|_p \ .$$

The function $g \varepsilon C_c(R^1)$, and hence there is a compact interval $[a,b]$ such that $g(x) = 0$ for all $x \notin [a,b]$. As noted in Lemma 2.18, g is uniformly continuous on R^1. Given $\varepsilon_1 > 0$, there exists a number δ, $0 < \delta < 1$, such that

$$(5)\quad |g_h(x) - g(x)| = |g(x + h) - g(x)| < \varepsilon_1$$

whenever $|h| < \delta$. Now consider $|h| < \delta$, and note that

$$\int_{-\infty}^{\infty} |g_h(x) - g(x)|^p \, dx = \int_{-\infty}^{a-1} |g_h(x) - g(x)|^p dx + \int_{a-1}^{b+1} |g_h(x) - g(x)|^p dx$$

$$+ \int_{b+1}^{\infty} |g_h(x) - g(x)|^p dx \, .$$

It follows, since $|h| < \delta < 1$, that $g_h(x) = 0 = g(x)$ for x in $(-\infty, a - 1] \cup [b + 1, \infty)$, since for example if $x \geq b + 1$, then $x + h \geq b$, and thus $g(x + h) = 0$. Using (5), we then find

$$(6) \quad \int_{-\infty}^{\infty} |g_h(x) - g(x)|^p dx = \int_{a-1}^{b+1} |g_h(x) - g(x)|^p dx < \varepsilon_1{}^p (b - a + 2) \, ,$$

if $|h| < \delta$. Now suppose we choose

$$\varepsilon_1 = \frac{\varepsilon}{3(b - a + 2)^{\frac{1}{p}}}$$

and use a corresponding δ in the above argument. In this case, (6) yields

$$\|g_h - g\|_p < \frac{\varepsilon}{3} \, ,$$

for $|h| < \delta$, and hence, by (4),

$$\|f_h - f\|_p < \varepsilon \, ,$$

for $|h| < \delta$. Thus

$$\lim_{h \to 0} \|f_h - f\|_p = 0 \ . \qquad \square$$

In a metric space, a compact set is necessarily closed and bounded. In particular this is the case in a normed linear space, where it is understood that the metric is that induced by the norm. For example, in $L^p(\mu)$, the metric d is defined by $d(f,g) = \|f - g\|_p$. Note again that, as applied to functions, d is not a metric. We agree as before to think of f and g as equivalence classes as defined in the discussion of L^p spaces, when we speak of the metric space $L^p(\mu)$. The fact that a closed and bounded set in a metric space need not be compact has already been illustrated by an example in the Banach space $C^r[0,1]$. It is known, for instance, that if S is the closed unit ball, $S = \{x : \|x\| \leq 1\}$, in a Banach space X, then S is a compact set in X if and only if the space X is of finite dimension as a linear space. This is proved in Yosida [13], p. 85, for example. An important compactness criterion for sets in L^p spaces is contained in the following theorem, for which more detailed references are given following the proof.

Theorem 2.20 (Riesz-Kolmogorov). Let $L^p(m)$ be the L^p space associated with Legesgue measure m in R^1, and suppose $1 \leq P < \infty$. A subset F of $L^p(m)$ is relatively compact in the metric space $L^p(m)$ if and only if F satisfies the following three conditions:

(a) $\sup \{\|f\|_p : f \varepsilon F\} < \infty$.

(b) For each $\varepsilon > 0$, there exists number $\delta > 0$ such that

$$\sup\{\|f_h - f\|_p : f \varepsilon F\} \leq \varepsilon$$

if $|h| < \delta$, and $f_h(x) = f(x + h)$ for all real x.

(c) For each $\varepsilon > 0$, there exists a compact interval K in R^1 such that

$$\sup\{\int_{R^1 - K} |f|^p dm : f \varepsilon F\} \leq \varepsilon \ .$$

Proof. Suppose F is relatively compact in $L^p(m)$. Then the closure $\bar{F}$ is compact, and consequently F is bounded in $L^p(m)$. This gives condition (a).

Let $\varepsilon > 0$ be given. Since $\bar{F}$ is compact, then by Theorem 2.12 F is totally bounded, and hence there is an $\frac{\varepsilon}{2}$-net, $\{f_1, f_2, \cdots, f_n\}$, in F. Thus for each $f \varepsilon F$ there exists an element f_j of the $\frac{\varepsilon}{2}$-net such that $\|f - f_j\|_p < \frac{\varepsilon}{2}$. Using the approximation theorem which asserts that $C_c(R^1)$, the space of continuous functions of compact support defined on R^1, is dense in $L^p(m)$, there exist functions $g_1, \cdots, g_n$ in $C_c(R^1)$ such that

$$(1) \quad \|f_j - g_j\|_p < \frac{\varepsilon}{2} \qquad (j = 1, 2, \cdots, n) \ .$$

Since each g_j has compact support and there are only finitely many functions g_j concerned here, there exists a compact interval K centered at $x = 0$ such that the functions $\{g_1, \cdots, g_n\}$ vanish uniformly on $R^1 - K$. If χ is the characteristic function of $R^1 - K$, it follows from Minkowski's inequality (Royden [9], pp. 95-96) and the identity

$f\chi = (f - g_j)\chi + g_j\chi$ that

$$(2) \quad \|f\,\chi\|_p \le \|(f - g_j)\chi\|_p + \|g_j\chi\|_p \ .$$

Now

$$(3) \quad \|g_j\chi\|_p = 0$$

since g_j vanishes on $R^1 - K$. Thus, for each $f\varepsilon F$, by (1), (2) and (3), and Minkowski's inequality,

$$(4) \quad \int_{R^1-K} |f|^p dm = \|f\chi\|_p \le \|(f - g_j)\chi\|_p$$

$$\le \|f - g_j\|_p$$

$$\le \|f - f_j\|_p + \|f_j - g_j\|_p < \varepsilon$$

if the index j is such that $\|f - f_j\|_p < \frac{\varepsilon}{2}$. This gives condition (c).

It remains to verify condition (b). Let $\varepsilon > 0$ be given, and let $\{f_1,\cdots,f_n\}$ be an $\frac{\varepsilon}{3}$ -net for F. If $f\varepsilon F$, then Minkowski's inequality implies that

$$(5) \quad \|f_h - f\|_p \le \|f_h - (f_j)_h\|_p + \|(f_j)_h - f_j\|_p + \|f_j - f\|_p$$

for each $j = 1,\cdots,n$. Since, by a simple change of variable formula for the Lebesgue integral (depending only on translation invariance of

Lebesgue measure in the real line, as observed in the proof of Lemma 2.19),

$$(6)\quad \int_{-\infty}^{\infty} |f(x + h) - f_j(x + h)|^p dx = \int_{-\infty}^{\infty} |f(x) - f_j(x)|^p dx ,$$

it follows that

$$(7)\quad \|f_h - f\|_p \leq 2\|f_j - f\|_p + \|(f_j)_h - f_j\|_p .$$

By Lemma 2.19, for each f_j there exists a $\delta_j > 0$ such that, if $|h| < \delta_j$,

$$\|(f_j)_h - f_j\|_p < \frac{\varepsilon}{3} .$$

If $\delta = \min\{\delta_1,\cdots,\delta_n\}$, then $|h| < \delta$ implies that

$$(8)\quad \|(f_j)_h - f_j\|_p < \frac{\varepsilon}{3} \qquad (j = 1,\cdots,n) .$$

Given $f\varepsilon F$, there exists an index j such that $\|f - f_j\|_p < \frac{\varepsilon}{3}$, and hence it follows from (7) and (8) that

$$(9)\quad \|f_h - f\|_p < 2\,\frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

if $|h| < \delta$. Note that this $\delta > 0$ suffices for all $f\varepsilon F$. Hence condition (b) is verified. This completes the proof of the necessity of conditions (a), (b), and (c) of the theorem.

Now suppose that F is a subset of $L^p(m)$, and that the conditions (a), (b), and (c) hold for F. For each real number x, define the *mean value*

$$(10)\quad \bar{f}_a(x) = \frac{1}{2a}\int_{-a}^{a} f_t(x)dt = \frac{1}{2a}\int_{x-a}^{x+a} f(t)dt \qquad (a > 0)\ ,$$

where f_t is again the translate of f defined by $f_t(x) = f(x + t)$. By Hölder's inequality (Royden [9], p. 95) applied to the functions $|f|$ and 1 on $[x - a, x + a]$, it follows that

$$(11)\quad \int_{x-a}^{x+a} |f(t)|dt \le \|f\|_p \,(2a)^{\frac{1}{q}} < \infty\ ,$$

where q is the index conjugate to p : $\frac{1}{p} + \frac{1}{q} = 1$ (with $q = \infty$ if $p = 1$). If $p = 1$ and hence $q = \infty$, then the bound is simply $\|f\|_p$. Thus the mean value function $\bar{f}_a$ is defined for each fixed $a > 0$. For each x, the number $\bar{f}_a(x)$ is the mean value of f on the interval $[x - a, x + a]$.

Now by (10)

$$\bar{f}_a(x) - f(x) = \frac{1}{2a}\int_{-a}^{a} (f(x + t) - f(x))dt$$

and thus

$$\|\bar{f}_a - f\|_p = \frac{1}{2a}\left(\int_{-\infty}^{\infty} \left|\int_{-a}^{a}(f(x + t) - f(x))dt\right|^p dx\right)^{\frac{1}{p}}$$

$$\le \frac{1}{2a}\left[\int_{-\infty}^{\infty} \left(\int_{-a}^{a} |f(x + t) - f(x)|dt\right)^p dx\right]^{\frac{1}{p}} .$$

Applying Hölder's inequality on the inner integral, it follows that

$$(12)\quad \|\bar{f}_a - f\|_p \le \frac{1}{2a}\left\{\int_{-\infty}^{\infty}\left[\left(\int_{-a}^{a}|f(x+t)-f(x)|^p dt\right)^{\frac{1}{p}}(2a)^{\frac{1}{q}}\right]^p dx\right\}^{\frac{1}{p}}$$

$$\le \frac{(2a)^{\frac{1}{q}}}{2a}\left[\int_{-\infty}^{\infty}\left(\int_{-a}^{a}|f(x+t)-f(x)|^p dt\right)dx\right]^{\frac{1}{p}} .$$

Since the function $h(x,t) = |f(x + t) - f(x)|^p$ is a non-negative Lebesgue measurable function of (x,t) on the strip $(-\infty,\infty) \times [-a,a]$ in the (x,t) - plane, Fubini's theorem (Royden [9], pp. 233-234) asserts that both orders of integration yield the double integral of h over the strip. Thus, changing the order of integration, it follows from (12) that

$$(13)\quad \|\bar{f}_a - f\|_p \le \frac{1}{(2a)^{\frac{1}{p}}}\left[\int_{-a}^{a}\left(\int_{-\infty}^{\infty}|f_t(x)-f(x)|^p dx\right)dt\right]^{\frac{1}{p}}$$

$$\le \frac{1}{(2a)^{\frac{1}{p}}}\left[\int_{-a}^{a}\|f_t - f\|_p^p dt\right]^{\frac{1}{p}} .$$

By condition (b), corresponding to $\varepsilon > 0$, there exists a number $\delta > 0$ such that, if $|t| < \delta$, then

$$(14)\quad \|f_t - f\|_p < \varepsilon$$

uniformly for $f \varepsilon F$. Hence, if $0 < a < \delta$ then (13) and (14) imply that

$$(15) \quad \|\bar{f}_a - f\|_p \le (2a)^{-\frac{1}{p}} \left(\int_{-a}^{a} \varepsilon^p dt\right)^{\frac{1}{p}} = \varepsilon ,$$

uniformly for $f \varepsilon F$. It has thus been established that

$$(16) \quad s - \lim_{a \to 0^+} \bar{f}_a = f$$

uniformly for $f \varepsilon F$, the limit concerned being the strong limit in $L^p(m)$, i.e., the limit in the $L^p(m)$ metric.

We now examine the set $M = \{\bar{f}_a : f \varepsilon F\}$ for a fixed $a > 0$. Note that $M \subset L^p(m)$.

For any real numbers x and x_o,

$$|\bar{f}_a(x) - \bar{f}_a(x_o)| = \frac{1}{2a} \left|\int_{-a}^{a} [f(x + t) - f(x_o + t)]dt\right|$$

$$\le \frac{1}{2a} \int_{-a}^{a} |f(x + t) - f(x_o + t)|dt .$$

Hölder's inequality implies that

$$|\bar{f}_a(x) - \bar{f}_a(x_o)| \le (2a)^{-\frac{1}{p}} \left(\int_{-a}^{a} |f(x + t) - f(x_o + t)|^p dt\right)^{\frac{1}{p}}$$

$$\le (2a)^{-\frac{1}{p}} \left(\int_{-\infty}^{\infty} |f(x + t) - f(x_o + t)|^p dt\right)^{\frac{1}{p}}$$

$$\le (2a)^{-\frac{1}{p}} \left(\int_{-\infty}^{\infty} |f(x - x_o + s) - f(s)|^p ds\right)^{\frac{1}{p}} .$$

Hence

$$|\bar{f}_a(x) - \bar{f}_a(x_o)| \le (2a)^{-\frac{1}{p}} \|f_{x-x_o} - f\|_p .$$

If $\varepsilon > 0$ is given, condition (b) implies that there exists a number $\delta > 0$ such that if $|x - x_o| < \delta$ then

$$|\bar{f}_a(x) - \bar{f}_a(x_o)| < \varepsilon$$

uniformly for $f\varepsilon F$. Thus the set $M = \{\bar{f}_a : f\varepsilon F\}$ is equicontinuous for each fixed $a > 0$. It will now be shown that M is also uniformly bounded for each fixed $a > 0$. The estimate in (11) shows that for every real number x,

$$|f_a(x)| \le (2a)^{-\frac{1}{p}} \|f\|_p \le (2a)^{-\frac{1}{p}} \sup\{\|f\|_p : f\varepsilon F\} < \infty ,$$

by condition (a). Thus M is, for each fixed $a > 0$, uniformly bounded and equicontinuous.

Let $\varepsilon_1 > 0$ be given, and choose $a > 0$ sufficiently small such that

$$(17) \quad \|\bar{f}_a - f\|_p < \frac{\varepsilon_1}{2} ,$$

uniformly for $f\varepsilon F$. This is possible because of (16), which asserts that $s - \lim_{a\to 0^+} \bar{f}_a = f$ uniformly for $f\varepsilon F$. Using condition (c) of the theorem, choose $\alpha > 0$ sufficiently large that, with $K = [-\alpha,\alpha]$,

$$(18) \quad \int_{R^1-K} |f|^p dx < \frac{\varepsilon_1{}^p}{2^{p+1}}$$

for every $f \varepsilon F$.

The set M, with the functions considered on the compact interval K, is a uniformly bounded and equicontinuous subset of $C^r(K)$, the Banach space of real-valued continuous functions on K with the uniform norm. It follows by the Arzelà-Ascoli theorem that M is relatively compact in $C^r(K)$, and thus is totally bounded by Theorem 2.12. Corresponding to $\varepsilon_3 = \frac{\varepsilon_1}{(2\alpha)^{\frac{1}{p}}} > 0$, there exists an $\frac{\varepsilon}{3}$-net, $\{\bar{f}_{1,a}, \cdots, f_{n,a}\}$. For each $f \varepsilon F$, there exists an index j such that

$$(19) \quad \|\bar{f}_a - \bar{f}_{j,a}\|_\infty = \sup\{|\bar{f}_a(x) - \bar{f}_{j,a}(x)| : x \varepsilon K\} < \varepsilon_3 .$$

We now establish the estimate of the $L^p(m)$ norm $\|f - f_j\|_p$ for the same index j. By the Minkowski inequality, and (17),

$$(20) \quad \|f - f_j\|_p \leq \|f - \bar{f}_a\|_p + \|\bar{f}_a - \bar{f}_{j,a}\|_p + \|\bar{f}_{j,a} - f_j\|_p$$

$$< \varepsilon_1 + \|\bar{f}_a - \bar{f}_{j,a}\|_p .$$

Now, from (19),

$$(21)\quad \|\bar{f}_a - \bar{f}_{j,a}\|_p^p = \int_{-\alpha}^{\alpha} |\bar{f}_a - \bar{f}_{j,a}|^p dx + \int_{R^1-K} |\bar{f}_a - \bar{f}_{j,a}|^p dx$$

$$< 2\alpha\varepsilon_3{}^p + \int_{R^1-K} |\bar{f}_a - f_{j,a}|^p dx \ .$$

By the Minkowski inequality and (17), it follows that

$$\Big(\int_{R^1-K} |\bar{f}_a - \bar{f}_{j,a}|^p dx\Big)^{\frac{1}{p}} = \Big(\int_{-\infty}^{\infty} |\bar{f}_a - \bar{f}_{j,a}|^p \chi_{R^1-K} dx\Big)^{\frac{1}{p}}$$

$$\leq \|\bar{f}_a - f\|_p + \|(f - f_j)\chi_{R^1-K}\|_p + \|f_j - \bar{f}_{j,a}\|_p$$

$$< \varepsilon_1 + \Big(\int_{R^1-K} |f - f_j|^p dx\Big)^{\frac{1}{p}} < \varepsilon_1 + \Big(\int_{R^1-K} \big(|f|^p + |f_j|^p\big)\, dx\Big)^{\frac{1}{p}} \ .$$

Noting the choice of $K = [-\alpha,\alpha]$ in (18), it follows that

$$(22)\quad \Big(\int_{R^1-K} |\bar{f}_a - \bar{f}_{j,a}|^p dx\Big)^{\frac{1}{p}} < \varepsilon_1 + \Big(2^p\ 2\ \frac{\varepsilon_1{}^p}{2^{p+1}}\Big)^{\frac{1}{p}} = 2\varepsilon_1 \ .$$

It follows from (21), (22) and the choice of $\varepsilon_3 = \dfrac{\varepsilon_1}{(2\alpha)^{\frac{1}{p}}}$, that

$$\|\bar{f}_a - \bar{f}_{j,a}\|_p^p < 2\alpha\ \frac{\varepsilon_1{}^p}{2\alpha} + (2\varepsilon_1)^p = (1 + 2^p)\varepsilon_1^p \ .$$

Thus, noting (20),

$$(23)\quad \|f - f_j\|_p < \varepsilon_1 + (1 + 2^p)^{\frac{1}{p}}\ \varepsilon_1 = \varepsilon_1[(1 + 2^p)^{\frac{1}{p}} + 1] \ .$$

The estimate in (23) holds for each $f \varepsilon F$ for a suitable index j. Thus, given $\varepsilon > 0$, there exists an ε-net in F. One has only to choose, in the preceding argument,

$$\varepsilon = \varepsilon_1[1 + (1 + 2^p)^{\frac{1}{p}}] .$$

Since the space $L^p(m)$ is complete, then Theorem 2.12 implies that $\bar{F}$ is compact. Thus F is relatively compact in $L^p(m)$, assuming the conditions (a), (b), and (c) of the theorem. $\square$

The criteria for relative compactness of a subset of $L^p(m)$, with $1 \leq p < \infty$, resemble those of the Arzelà-Ascoli theorem. Thus condition (a) requires uniform boundedness of the subset F in the $L^p(m)$ metric, and condition (b) is an equicontinuity condition in the $L^p(m)$ metric. Condition (c) asserts that the functions F are "equivanishing at ∞" in the $L^p(m)$ metric. Versions of the theorem in R^1 with $1 \leq p < \infty$ are due to M. Riesz [8], M. Fréchet [2], and A. Kolmogorov [6]. The theorem has been extended to the case that $0 < p < 1$ by M. Tsuji [11]. The proof in this thesis is a greatly expanded and somewhat simplified version of a proof given by K. Yosida [13]. A far-reaching generalization of the theorem to the case of translation invariant measures on locally compact topological groups is given by A. Weil [12].

CHAPTER III

TIETZE EXTENSION THEOREMS

If a continuous function is defined on a subset of a metric space, then the question arises as to whether this function can be extended continuously to the whole space. We note that this is not always the case by letting $X = R^1$, and let D consist of all real $x \neq 0$, and $f(x) = \frac{x}{|x|}$ for x in D. It is the purpose of this chapter to prove two versions of an extension theorem due to Tietze [10]. To this end, several preliminary results are required.

Definition 3.1. A function g is said to be an *extension* of a function f if the domain X of g contains the domain D of f, and $g(x) = f(x)$ for all x in D.

Lemma 3.2. Let D be a non-empty closed subset of a metric space X. If $x \in X$, let $d(x) = d(x,D)$ be the distance from x to D,

$$d(x,D) = \inf\{d(x,y) : y \in D\} .$$

Then, (1) $d(x)$ is non-negative on X.

(2) $d(x) = 0$ if and only if $x \in D$.

(3) $d(x)$ is a continuous function on X.

Proof. Parts (1) and (2) follow from the definition of $d(x,D)$, and the fact that D is closed. (In general, $d(x) = 0$ if and only if $x \varepsilon \bar{D}$, the closure of D.)

To prove part (3), let x and y be fixed in X. Now $d(x,D) \leq d(x,z)$ for every z in D by the definition of $d(x,D)$. Since $d(x,z) \leq d(x,y) + d(y,z)$, then

$$d(x,y) + d(y,z) \geq d(x,D)$$

for every z in D. Thus

$$d(y,z) \geq d(x,D) - d(x,y)$$

for every z in D. It follows from the definition of $d(y,D)$ that

$$d(y,D) \geq d(x,D) - d(x,y) .$$

Hence

$$d(x,y) \geq d(x,D) - d(y,D) .$$

By a similar argument, with x and y interchanged, it follows that

$$d(x,y) \geq d(y,D) - d(x,D) .$$

Thus

$$d(x,y) \geq |d(x,D) - d(y,D)| .$$

Now let $\varepsilon > 0$ be given, and let $\delta = \varepsilon$. Then $d(x,y) < \delta$ implies that $|d(x) - d(y)| < \varepsilon$. Thus $d(x) = d(x,D)$ is a continuous function on X.

$\square$

Lemma 3.3. If A and B are disjoint non-empty closed subsets of a metric space X, then there is a continuous function f mapping X into R^1 such that $|f(x)| \leq 1$ for all x in X, $f(x) = 1$ for all x in A, and $f(x) = -1$ for all x in B.

Proof. For each $x \varepsilon X$, let $d_1(x) = d(x,A)$ and $d_2(x) = d(x,B)$. It follows from Lemma 3.2 that $d_1(x)$ and $d_2(x)$ are continuous and their sum is positive, since A and B are disjoint closed sets. Now define the function f such that

$$f(x) = \frac{d_2(x) - d_1(x)}{d_1(x) + d_2(x)}$$

for every x in X. It follows that

$$f(x) = \frac{d_2(x) - d_1(x)}{d_1(x) + d_2(x)} = \frac{d_2(x)}{d_2(x)} = 1$$

if $x \varepsilon A$ and

$$f(x) = \frac{d_2(x) - d_1(x)}{d_1(x) + d_2(x)} = - \frac{d_1(x)}{d_1(x)} = -1$$

if $x \varepsilon B$. Now f is a continuous function since d_1 and d_2 are continuous functions and $d_1(x) + d_2(x) > 0$. Now, for all x in X,

$$|f(x)| = \left|\frac{d_2 - d_1}{d_1 + d_z}\right| \leq 1 ,$$

and f has the properties required. ▯

Lemma 3.4. If f is a continuous function mapping a closed non-empty subset D of a metric space X into R^1, and $|f|$ has a finite bound M on D, then there exists a continuous function g mapping X into R^1 such that $|g| \leq \frac{M}{3}$ on X and $|g(x) - f(x)| \leq \frac{2M}{3}$ for all x in D.

Proof. Suppose $f(x) \geq -\frac{M}{3}$ for all $x \varepsilon D$. Define $g(x) = \frac{M}{3}$ for all $x \varepsilon X$. Then g is a continuous function, g maps X into R^1 and $|g| = \frac{M}{3} \leq \frac{M}{3}$. Since $|f| \leq M$ for all $x \varepsilon X$, then $-\frac{M}{3} \leq f(x) \leq M$ for all $x \varepsilon D$ and thus

$$-\frac{M}{3} - \frac{M}{3} \leq f(x) - g(x) \leq M - \frac{M}{3} .$$

Hence

$$|g(x) - f(x)| \leq \frac{2M}{3}$$

for all $x \varepsilon D$.

Now suppose $f(x) \leq \frac{M}{3}$ for all $x \varepsilon D$. Define $g(x) = -\frac{M}{3}$. As before, it follows that g is a continuous function, g maps X into R^1 and $|g| = \frac{M}{3} \leq \frac{M}{3}$. Since $|f| \leq M$ for all $x \varepsilon X$, then $-M \leq f(x) \leq \frac{M}{3}$ for all $x \varepsilon D$ and thus

$$-M + \frac{M}{3} \leq f(x) - g(x) \leq \frac{M}{3} + \frac{M}{3} .$$

Hence

$$|g(x) - f(x)| \leq \frac{2M}{3}$$

for all $x \varepsilon D$.

Otherwise, the sets $A = \{x : x\varepsilon D,\ f(x) \geq \frac{M}{3}\}$ and $B = \{x : x\varepsilon D,\ f(x) \leq -\frac{M}{3}\}$ are closed and non-empty. We note that $A \cap B = \phi$. Thus the previous lemma applies, and there exists a continuous function H mapping X into R^1 such that $|H(x)| \leq 1$ for all $x\varepsilon X$, $H(x) = 1$ for all $x\varepsilon A$, and $H(x) = -1$ for all $x\varepsilon B$. Define

$$g(x) = \frac{MH(x)}{3}$$

for all $x\varepsilon X$. Then

$$|g(x)| = |\frac{M}{3} H(x)| = |\frac{M}{3}|\,|H(x)| \leq |\frac{M}{3}| = \frac{M}{3}$$

for all $x\varepsilon X$. If $x\varepsilon A$, then

$$g(x) = \frac{MH(x)}{3} = \frac{M}{3} \leq f(x) \leq M .$$

Thus

$$0 \leq f(x) - g(x) \leq \frac{2M}{3}$$

for all $x \varepsilon A$. If $x \varepsilon B$, then

$$g(x) = \frac{M}{3} H(x) = -\frac{M}{3} \geq f(x) \geq -M .$$

Thus

$$0 \geq f(x) - g(x) \geq -\frac{2M}{3}$$

for all $x \varepsilon B$. If $x \varepsilon D - (A \cup B)$, then $x \notin A$, $x \notin B$ and thus $-\frac{M}{3} \leq f(x) \leq \frac{M}{3}$. That is, $|f(x)| \leq \frac{M}{3}$ for all $x \varepsilon D - (A \cup B)$. Since

$$|g(x)| \leq \frac{M}{3} \text{ for all } x \varepsilon X, \text{ then}$$

$$|g(x) - f(x)| \leq |f(x)| + |g(x)| \leq \frac{M}{3} + \frac{M}{3} = \frac{2M}{3}$$

for all $x \varepsilon D - (A \cup B)$. In each case $|g(x) - f(x)| \leq \frac{2M}{3}$ and thus $|g(x) - f(x)| \leq \frac{2M}{3}$ for all $x \varepsilon D$. ▯

Lemma 3.5. Any closed interval $[a,b] \subset R^1$, with $a < b$, can be linearly mapped onto $[-1,1]$. Any bounded open interval $(a,b) \subset R^1$, with $a < b$, can be linearly mapped onto $(-1,1)$.

Proof. Consider the closed interval $[a,b] \subset R^1$ with $a < b$. Define

$$H(x) = \frac{a + b - 2x}{b - a}$$

for all $x \varepsilon [a,b]$. Then $H(b) = -1$, $H(a) = 1$, and since H is continuous and decreasing on $[a,b]$, H takes on all values between -1 and 1.

The same function H, restricted to (a,b), yields the second assertion of the lemma. ▯

Definition 3.6. A one-to-one mapping f of X onto Y is called a *homeomorphism* between X and Y if f is continuous and if f^{-1} exists and is also continuous.

Lemma 3.7. R^1 is homeomorphic with the open interval $(-1,1)$ in R^1.

Proof. Define

$$f(x) = \frac{x}{1 + |x|}$$

for $x \varepsilon R^1$. Then f is continuous and

$$|f(x)| = \left|\frac{x}{1 + |x|}\right| < 1$$

for all $x \varepsilon R^1$. Now suppose that $f(x) = f(y)$, then $f(|x|) = f(|y|)$ and thus

$$\frac{|x|}{1 + |x|} = \frac{|y|}{1 + |y|} \quad .$$

Then $|x| = |y|$ and it follows from the definition of f that $x = y$. This implies that f is a one-to-one mapping of R^1 onto $(-1,1)$ and thus f has an inverse. It follows readily that

$$f^{-1}(x) = \frac{x}{1 - |x|}$$

for $x\varepsilon(-1,1)$ and this function is continuous. Thus R^1 is homemorphic with $(-1,1)$. ▯

The proof of the following theorem follows the method used in McShane and Botts [7].

Theorem 3.8 (Tietze Extension Theorem, Version I). Let D be a closed non-empty subset of a metric space X, and let S be either a closed interval in R^1, an open interval in R^1, or R^1 itself. Let f be a continuous mapping of D into S. Then f has an extension g which maps X into S and g is continuous on X.

Proof. Suppose that S is the closed interval $[-1,1]$. Then $|f(x)| \leq 1$ for all $x\varepsilon D$, and thus Lemma 3.4 implies there is a continuous function f_1 mapping X into R^1 such that $|f_1| \leq \frac{1}{3}$ on X and $|f(x) - f_1(x)| \leq \frac{2}{3}$ for all $x\varepsilon D$. Now apply Lemma 3.4 to $f_1 - f$ on D. We obtain a continuous function f_2 mapping X into R^1 such that $|f_2| \leq \frac{1}{3}(\frac{2}{3})$ on X and $|f(x) - f_1(x) - f_2(x)| < (\frac{2}{3})^2$ for all $x\varepsilon D$. Proceeding inductively, for each positive integer n we obtain a continuous function f_n mapping X into R^1 such that $|f_n| \leq \frac{1}{3}(\frac{2}{3})^{n-1}$ on X and $|f(x) - f_1(x) - \cdots - f_n(x)| < (\frac{2}{3})^n$ for all $x\varepsilon D$, and each $n \geq 1$. Noting that

$$0 \leq |f_n(x)| \leq \frac{1}{3}\left(\frac{2}{3}\right)^{n-1} \qquad (n = 1,2,\cdots)$$

and for all $x \varepsilon X$, it follows from the Weierstrass M-test that $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on X. Let

$$g(x) = \sum_{i=1}^{\infty} f_i(x)$$

for each $x \varepsilon X$. Since each f_i is continuous and the series converges uniformly on X, then g is continuous on X. Further,

$$|g(x)| = \left|\sum_{i=1}^{\infty} f_i(x)\right| \leq \sum_{i=1}^{\infty} |f_i(x)|$$

$$\leq \sum_{i=1}^{\infty} \frac{1}{3} \left(\frac{2}{3}\right)^{i-1} = 1$$

and hence $|g(x)| \leq 1$ for all $x \varepsilon X$.

Since

$$0 \leq |f(x) - f_1(x) - \cdots - f_n(x)| \leq \left(\frac{2}{3}\right)^n$$

for all $x \varepsilon D$, it follows that

$$\lim_{n\to\infty} (f(x) - f_1(x) - \cdots - f_n(x)) = 0$$

for all $x \varepsilon D$. Thus

$$f(x) - \lim_{n\to\infty} \sum_{i=1}^{n} f_i(x) = 0 .$$

Hence

$$f(x) = \sum_{i=1}^{\infty} f_i(x) = g(x)$$

for all $x \varepsilon D$. This completes the proof if S is [-1,1]. Since any closed interval can be linearly mapped onto [-1,1] by Lemma 3.6, this completes the proof for S a compact interval.

Now suppose that S is the open interval (-1,1). By the preceding results, f has an extension f_1 mapping X into [-1,1] with f_1 continuous on X since $f(D) \subset (-1,1) \subset [-1,1]$. For each $x \varepsilon X$, let d(x) be the distance from x to D. This function is zero if $x \varepsilon D$, positive if $x \notin D$, and it is continuous on X by Lemma 3.2. Define

$$g(x) = f_1(x)[\max\{0,(1 - d(x))\}]$$

for all $x \varepsilon X$. Then g is continuous since f_1, 0, 1, d(x) are all continuous, the difference of continuous functions is continuous, the maximum of two continuous functions is continuous, and the product of two continuous functions is continuous. If $x \varepsilon D$ then $d(x) = 0$ and $\max\{0,(1 - d(x))\} = 1$, and thus $g(x) = f_1(x)$. If $x \notin D$ then $|f_1(x)| \leq 1$ since f_1 maps X into [-1,1] and $\max\{0,(1 - d(x))\} < 1$. Thus

$$|g(x)| = |f_1(x)[\max\{0,(1 - d(x))\}]|$$

$$= |f_1(x)|\ |[\max\{0,(1 - d(x))\}]| < 1$$

if $x \notin D$. Hence $|g(x)| < 1$ for all $x \in X$, and the proof is complete if S is $(-1,1)$. Since any open interval can be mapped linearly onto $(-1,1)$ by Lemma 3.6, this completes the proof whenever S is an open interval.

Now suppose that S is R^1. Then Lemma 3.7 implies there is a homeomorphic mapping H which maps R^1 onto $(-1,1)$. The composite function $h = H \circ f$ is continuous on D since f is continuous on $D \subset X$ and H is continuous on $f(D) \subset R^1$. The range of h lies in $(-1,1)$ since the range of H is $(-1,1)$. Thus h has an extension h_1 mapping X into $(-1,1)$ with h_1 continuous on X by the second part of this theorem. Now define the composite function $g = H^{-1} \circ h_1$. It follows that g is continuous on X since h_1 is continuous on X and H^{-1} is continuous on $h_1(X) \subset (-1,1)$. If $x \in D$, then

$$g(x) = H^{-1}\big(h_1(x)\big) = H^{-1}\big(H(f(x))\big) = f(x) \;,$$

and the proof is complete if S is R^1. $\square$

The proof of the following version follows the method in Dieudonné [1].

Theorem 3.9 (Tietze Extension Theorem, Version II). Let X be a metric space, D a closed non-empty subset of X, f a bounded continuous function mapping D into R^1. Then there exists a continuous function g mapping X into R^1 which coincides with f on D and is such that

$$\sup_{x \in X} g(x) = \sup_{y \in D} f(y), \quad \inf_{x \in X} g(x) = \inf_{y \in D} f(y) \;.$$

Proof. Since f is a bounded continuous function mapping D into R^1, let $m \leq f(y) \leq M$ on all $y \varepsilon D$. Suppose $m = M$, then $f(y) = M$ for all $y \varepsilon D$. In this case, let $g(x) = M$ for all $x \varepsilon X$ and the theorem is true.

Now suppose that $M > m$. Let $\alpha = \frac{1}{M - m} > 0$ and $\beta = \frac{M - 2m}{M - m}$. Then

$$\alpha m + \beta \leq \alpha f(y) + \beta \leq \alpha M + \beta$$

for all $y \varepsilon D$. From the choice of α and β, it follows that

$$1 \leq \alpha f(y) + \beta \leq 2$$

for all $y \varepsilon D$. Thus given any bounded continuous function mapping D into R^1, we can replace this function by a bounded continuous function mapping D into R^1 which is bounded below by 1 and above by 2. Hence, without loss of generality, we may assume that the given function f is such that

$$\inf_{y \varepsilon D} f(y) = 1, \quad \sup_{y \varepsilon D} f(y) = 2 .$$

Now define the function g such that

$$g(x) = \begin{cases} f(x) & \text{if } x \varepsilon D \\ \dfrac{\inf_{y \varepsilon D} (f(y) d(x,y))}{d(x,D)} & \text{if } x \varepsilon X - D . \end{cases}$$

Since, by definition,

$$d(x,D) = \inf_{y\varepsilon D} d(x,y) ,$$

and $1 \leq f(y) \leq 2$ for $y\varepsilon D$, then

$$d(x,y) \leq f(y)d(x,y) \leq 2d(x,y)$$

for $y\varepsilon D$ since $d(x,y) > 0$ for all $y\varepsilon X - D$ by Lemma 3.2. Thus

$$\inf_{y\varepsilon D} d(x,y) \leq \inf_{y\varepsilon D} f(y)d(x,y) \leq \inf_{y\varepsilon D} 2d(x,y) .$$

Hence,

$$\frac{\inf_{y\varepsilon D} d(x,y)}{d(x,D)} \leq \frac{\inf_{y\varepsilon D} f(y)d(x,y)}{d(x,D)} \leq \frac{\inf_{y\varepsilon D} 2d(x,y)}{d(x,D)}$$

and thus $1 \leq g(x) \leq 2$ for $x\varepsilon X - D$. The proof will be complete if we show that g is continuous at every point $x\varepsilon X$.

If $x\varepsilon D^{\circ}$ (the interior of D), then $g(x) = f(x)$ and the continuity of g on D° follows from the continuity of f on D° by hypothesis.

For x in the open set $X - D$,

$$g(x) = \frac{h(x)}{d(x,D)}$$

with

$$h(x) = \inf_{y\varepsilon D} (f(y)d(x,y)) .$$

Since $d(x,D)$ is continuous and $d(x,D) > 0$ for $x\varepsilon X - D$ by Lemma 3.2, it is only necessary to prove that h is continuous at every $x\varepsilon X - D$. Let $\varepsilon > 0$ be given, and let $r = d(x,D)$. If x' is such that

$$d(x,x') \le \frac{\varepsilon}{4} < r ,$$

then

$$d(x,y) \le d(x,x') + d(x',y) \le \frac{\varepsilon}{4} + d(x',y) .$$

Since

$$h(x') = \inf_{y\varepsilon D} (f(y)d(x',y))$$

for $x'\varepsilon X - D$, and $f(y) > 0$ since $1 \le f(y) \le 2$ for $y\varepsilon D$, then

$$f(y)d(x,y) \le \frac{\varepsilon}{4} f(y) + f(y)d(x',y) \le \frac{\varepsilon}{4} (2) + f(y)d(x',y)$$

and

$$\inf_{y\varepsilon D}(f(y)d(x,y)) \le \frac{\varepsilon}{2} + \inf_{y\varepsilon D}(f(y)d(x',y)) .$$

Thus

$$h(x) \leq \frac{\varepsilon}{2} + h(x') .$$

Similarly, since

$$d(x',y) \leq d(x',x) + d(x,y) \leq \frac{\varepsilon}{4} + d(x,y) ,$$

it follows that

$$h(x') \leq \frac{\varepsilon}{2} + h(x) .$$

Thus

$$|h(x') - h(x)| \leq \frac{\varepsilon}{2} < \varepsilon$$

for $d(x',x) \leq \frac{\varepsilon}{4} < r$. Thus h is continuous for $x \varepsilon X - D$ and, by a previous remark, it follows that g is continuous for $x \varepsilon X - D$.

Now suppose that x is a boundary point of D. Let $\varepsilon > 0$ be given, and let $r > 0$ be such that for $y \varepsilon D \cap B(x,r)$, where $B(x,r)$ by Definition 2.7 is a neighborhood of x of radius r,

$$(1) \quad |f(y) - f(x)| \leq \frac{\varepsilon}{2} .$$

Let $C = D \cap B(x,r)$ and $E = D - C$. If $x' \varepsilon X - D$ and $d(x',x) \leq \frac{r}{4}$, then for each $y \varepsilon E$,

$$(2) \quad d(x',y) + d(x,x') \geq d(x,y)$$

and

$$(3) \quad -d(x,x') \geq -\frac{r}{4} \ .$$

Now from (2) and (3) it follows that

$$(4) \quad d(x',y) \geq d(x,y) - d(x,x') \geq d(x,y) - \frac{r}{4} \geq r - \frac{r}{4} = \frac{3r}{4} \ ,$$

where $d(x,y) \geq r$, since if $y \varepsilon E$ then $y \notin C$ and thus $y \notin B(x,r)$. Thus

$$f(y)d(x',y) \geq \frac{3r}{4}$$

where $f(y) \geq 1$ for $y \varepsilon E$, since $y \varepsilon E$ implies $y \varepsilon D$ and $y \notin C$. Hence

$$(5) \quad \inf_{y \varepsilon E}(f(y)d(x',y)) \geq \frac{3r}{4} \ .$$

Since D is closed, and x is a boundary point of D, $x \varepsilon D$ and, since $f(x) \leq 2$ for x D, it follows that

$$(6) \quad f(x)d(x',x) \leq 2d(x',x) \leq 2 \ \frac{r}{4} = \frac{r}{2} \ .$$

Thus from (5) and (6) it follows that

$$(7)\quad \inf_{y\varepsilon D}(f(y)d(x',y)) = \inf_{y\varepsilon C}(f(y)d(x',y)) \,.$$

By (1),

$$(8)\quad f(x) - \frac{\varepsilon}{2} \leq f(y) \leq f(x) + \frac{\varepsilon}{2}$$

for $y\varepsilon C$, and by (4) it follows that

$$(9)\quad \inf_{y\varepsilon C} d(x',y) = \inf_{y\varepsilon D} d(x',y) = d(x',D) > 0 \,.$$

Thus by (4) and (8),

$$(f(x) - \frac{\varepsilon}{2})d(x',y) \leq f(y)d(x',y) \leq (f(x) + \frac{\varepsilon}{2})d(x',y) \,.$$

Hence by (7) and (9)

$$(f(x) - \frac{\varepsilon}{2})\inf_{y\varepsilon C} d(x',y) \leq \inf_{y\varepsilon C} f(y)d(x',y) \leq (f(x) + \frac{\varepsilon}{2})\inf_{y\varepsilon C} d(x',y)$$

and thus

$$(f(x) - \frac{\varepsilon}{2})d(x',D) \leq \inf_{y\varepsilon D}(f(y)d(x',y)) \leq (f(x) + \frac{\varepsilon}{2})d(x',D) \,.$$

Hence

$$f(x) - \frac{\varepsilon}{2} \leq \frac{\inf_{y\varepsilon D}(f(y)d(x',y))}{d(x',D)} \leq f(x) + \frac{\varepsilon}{2} \quad ,$$

and by the definition of $g(x)$ for $x \varepsilon X - D$, then

$$f(x) - \frac{\varepsilon}{2} \leq g(x') \leq f(x) + \frac{\varepsilon}{2}$$

for $x' \varepsilon X - D$. Thus

$$|g(x') - g(x)| = |g(x') - f(x)| \leq \frac{\varepsilon}{2} < \varepsilon$$

for $x' \varepsilon X - D$ and $d(x,x') \leq \frac{r}{4}$. If $x' \varepsilon D$ and $d(x,x') \leq \frac{r}{4}$ then

$$|g(x') - g(x)| = |f(x') - f(x)| < \varepsilon .$$

Thus g is continuous at each boundary point of D. By a previous remark, this completes the proof, since it has been shown that g is continuous at every point of X. $\square$

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